REFACTE · EXPOSITIONS

When a Formal Logarithm Fails

Corrections, cohomology, and a spectral sequence in characteristic three

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A logarithm would replace a multiplication law by addition. We will try to build that change of coordinates one coefficient at a time, using the error in the desired identity to choose the next coefficient. In characteristic three the quadratic repair succeeds and the cubic repair is impossible.

The first section performs that finite calculation directly. The later sections make the same repair process reusable: substitution records how errors compose, a complex distinguishes an error from an available correction, and a weight filtration keeps track of the stage being repaired. The spectral-sequence differential will be a description of the obstruction already calculated, with its independence from earlier choices proved.

The opening requires polynomial expansion and arithmetic in characteristic three. Tensor products and cochain complexes enter when they organise the correction problem. Section 7 gives a general lift-and-correct interpretation; section 8 puts the surviving polynomial to work in a different addition law. Only finitely many polynomial weights are used, so no convergence theorem for an infinite spectral sequence is needed.

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1 The identity and its obstruction

Let k be a field. Write a unit near 1 as $1 + t$. Multiplication gives the law

$$F(x, y) = (1 + x)(1 + y) - 1 = x + y + xy.$$

This is the multiplicative formal group in the coordinate t. Its identity is 0, its inverse is the formal series $-t/(1 + t)$, and associativity follows from associativity of multiplication. We seek a series

$$L(t) = t + \ell_2 t^2 + \ell_3 t^3 + \cdots$$

such that

$$L(F(x, y)) = L(x) + L(y).$$

The coefficient of t is prescribed to be 1. Such a series would be an invertible change of formal coordinate, turning the multiplicative law into the additive law.

1.1 Calculate the first two repairs

Before introducing a general language, write a candidate through cubic order:

$$L(t) = t + at^2 + bt^3 + O(t^4).$$

Use the coefficient of each total degree in the defect $E_L(x, y) = L(F(x, y)) - L(x) - L(y)$. Direct expansion gives

$$E_L(x, y) = (1 + 2a)xy + (2a + 3b)(x^2y + xy^2) + O((x, y)^4).$$

The quadratic error determines a, rather than asking us to guess it. Where 2 is invertible, $a = -1/2$. Where 3 is also invertible, the cubic error then gives $b = 1/3$. These are the first logarithm coefficients.

Now impose characteristic three. The first equation is $1 - a = 0$, so $a = 1$. The second coefficient is then -1, independent of b. More explicitly,

$$E_{t+t^2}(x, y) = -xy(x + y) + x^2y^2.$$

Adding bt^3 changes this defect by

$$b\big(F(x, y)^3 - x^3 - y^3\big) = bx^3y^3.$$

It cannot affect the cubic term. Nor can a term of degree four or higher affect a cubic coefficient. Thus a coordinate beginning with t can make the law additive through degree two, but not through degree three. We have both a successful repair and a proof of its first limit.

At each degree we ask which part of that error a new coefficient can change. The remaining sections prove that this coefficient procedure is consistent under associativity and identify its obstruction without depending on arbitrary choices of earlier repair.

In characteristic zero,

$$L(t) = \log(1+t) = \sum_{n\geq 1} \frac{(-1)^{n+1}}{n} t^n$$

does the job. One proof differentiates the proposed identity in x:

$$\frac{\partial}{\partial x}(L(F(x,y)) - L(x) - L(y)) = \frac{1+y}{1+F(x,y)} - \frac{1}{1+x} = 0.$$

The difference vanishes at $x = 0$, so it is zero. Here the last inference uses characteristic zero: a formal series with zero x-derivative is independent of x.

In characteristic p, $d(t^p)/dt = 0$, and that inference loses information. There is a second issue: a logarithm with the required normalization does not exist. The coefficients with denominator divisible by p signal an actual obstruction, not merely a poor formula.

For characteristic three, the obstruction has a short proof. The threefold sum under F is

$$[3]_F(t) = (1+t)^3 - 1 = t^3.$$

The threefold sum under the additive law is zero. Consequently any formal-group homomorphism L from the multiplicative group to the additive group satisfies

$$L(t^3) = 3L(t) = 0.$$

Substitution $t \mapsto t^3$ is injective on $k[[t]]$, so $L = 0$. In particular, there is no such homomorphism with linear coefficient 1.

This independently checks the nonexistence found in the coefficient repair. We now organise the successful quadratic step and the failed cubic step so that the residual has a well-defined place and is independent of choices. From now on, $\operatorname{char} k = 3$.

2 Turn substitution into an operation on functions

The repair used coefficients of substitutions, including coefficients that ordinary differentiation can miss. To calculate them directly, define the Hasse coefficients by

$$f(x+h) = \sum_{n\geq 0} D^{[n]} f(x) h^n, \qquad D^{[n]}(x^m) = \binom{m}{n} x^{m-n}.$$

This definition works in every characteristic, for polynomials or formal power series. For example, $D^{[3]}(x^3) = 1$, although the ordinary derivative of x^3 vanishes. The Frobenius calculation in *What a Derivative Retains* explains this distinction; *Divisibility of Separated Polynomials* uses these coefficients to retain multiplicities in a fiber.

The diagonal gives a geometric setting for these coefficients. Put $A = k[[t]]$. The completed tensor product is

$$A \widehat{\otimes}_k A = k[[x,y]].$$

The diagonal pullback $k[[x,y]] \to k[[t]]$ sends both x and y to t. Its kernel is $(y-x)$. Writing $h = y-x$, the successive quotients by powers of that ideal retain successive powers of displacement. The first

quotient by h^2 gives the usual first-order differential; the entire completion retains every Hasse coefficient.

Our identity asks for another compatibility as well: the coefficients must respect a group law. That law pulls functions in the other direction:

$$\Delta : A \longrightarrow A\widehat{\otimes}_k A, \qquad \Delta f = f(F(x,y)).$$

Thus the diagonal identifies two points, while Δ evaluates a function on their group product. In tensor notation,

$$\Delta t = t \otimes 1 + 1 \otimes t + t \otimes t.$$

For $g = 1 + t$, this is the simple identity

$$\Delta g = g \otimes g.$$

An element with constant term one satisfying this identity is called *group-like*. An element f satisfying

$$\Delta f = f \otimes 1 + 1 \otimes f$$

is called *primitive*. The requested logarithm would turn this group-like coordinate into a primitive series with linear coefficient 1.

The point of recording substitution as an operation is that associativity becomes an equation we can apply to any defect:

$$(\Delta \otimes 1)\Delta = (1 \otimes \Delta)\Delta.$$

Both sides send g to $g \otimes g \otimes g$. This coassociativity will ensure that failures of the primitive equation satisfy a compatibility equation of their own.

The same multiplicative law already occurs in the line-bundle calculation in *What a Derivative Retains*. In *Fourier, Symmetry, and Reconstruction*, §12.4, a logarithmic coordinate makes tensor products of unipotent monodromies additive over $\mathbf{C}$. Our calculation examines the obstruction to that universal coordinate change in characteristic three. It does not prohibit every logarithm calculation in a sufficiently small nilpotent quotient.

3 Organise defects and available corrections

The direct calculation tells us to compare the defect of a trial coordinate with the changes another coefficient can produce. We now build a linear bookkeeping system for those two jobs. Let $\mathfrak{m} = tk[[t]]$, and remove the two terms forced by the identity element:

$$\overline{\Delta} f = \Delta f - f \otimes 1 - 1 \otimes f, \qquad f \in \mathfrak{m}.$$

The remaining series vanishes when either input is zero. It therefore lies in $\mathfrak{m}\widehat{\otimes}\mathfrak{m}$. Its vanishing is exactly the primitive condition.

Define cochains by

$$C^0 = k, \qquad C^n = \mathfrak{m}^{\widehat{\otimes} n} \quad (n \geq 1).$$

Concretely, C^n consists of the series in $k[[x_1, \ldots, x_n]]$ divisible by $x_1 \cdots x_n$. They vanish whenever an input is the identity. Write a tensor as $f_1 \mid \cdots \mid f_n$. The differential is

$$D(f_1 \mid \cdots \mid f_n) = \sum_{i=1}^{n} (-1)^i f_1 \mid \cdots \mid \overline{\Delta} f_i \mid \cdots \mid f_n,$$

with $D = 0$ on C^0. Inserting $\overline{\Delta} f_i$ replaces one slot by two, so D raises cohomological degree by one. This is the normalized completed cobar complex of the coproduct.

For a one-cochain,

$$Df(x, y) = f(x) + f(y) - f(F(x, y)).$$

For a two-cochain,

$$Dh(x, y, z) = h(y, z) - h(F(x, y), z) + h(x, F(y, z)) - h(x, y).$$

The four terms compare the two ways of grouping three inputs. Coassociativity gives $D^2 = 0$: when the reduced coproduct is inserted twice, disjoint insertions cancel with opposite signs, and the two insertions in a single original slot agree by coassociativity and cancel as well. Thus every defect Df is automatically a two-cocycle.

Cochains can also be concatenated. If $u \in C^r$ and $v \in C^s$, place the slots of v after those of u. This associative product satisfies

$$D(uv) = D(u)v + (-1)^r u D(v).$$

It is not imposed to be graded commutative. In particular, $t \mid t$ need not vanish. The distinction between two kinds of multiplication is essential:

$$t^2 \in C^1, \qquad t \mid t \in C^2.$$

The first squares a function in one variable; the second puts that function in two slots. Both have polynomial weight two, but their cohomological degrees differ.

The sign convention gives $DL = -E_L$, so the requested identity is $DL = 0$. Adding a correction changes DL by its differential, and E_L by the negative of that differential. The complex therefore records exactly the operation we performed on coefficients in section 1; its square-zero identity checks that every trial error satisfies the required compatibility. The filtration will isolate which corrections are allowed at each stage.

4 Filtering by the order retained

Give the tensor monomial

$$t^{i_1} \mid \cdots \mid t^{i_n}$$

weight $i_1 + \cdots + i_n$, where each $i_j \geq 1$. Let F^sC^n consist of series whose monomials have weight at least s. On $C^0 = k$, take $F^0C^0 = k$ and $F^1C^0 = 0$.

Substitution by $x + y + xy$ cannot lower total degree. Consequently

$$D(F^sC) \subseteq F^sC.$$

The quotient $C/F^{s+1}C$ turns "ignore terms above degree s" into an exact finite operation. A series is a cocycle there when its identity holds through that order. Lifting it to the next quotient means choosing the next coefficient so that one further order also holds. Thus each quotient records a solution of the identity through a specified order.

The part of the coproduct that preserves weight is

$$\Delta_0 t = t \otimes 1 + 1 \otimes t.$$

Thus the associated graded complex uses the additive group law. The term $t \otimes t$, which distinguished multiplication from addition, raises weight. Let D_j denote the component of D that raises weight by j, so

$$D = D_0 + D_1 + D_2 + \cdots .$$

Here D_0 is the additive cobar differential. This decomposition uses the chosen coordinate and its homogeneous monomials; the filtration itself is the sequence of order cutoffs just defined.

Three gradings or indices now have separate roles. Cohomological degree counts input slots. Weight counts powers of the formal coordinates. The index of a spectral-sequence differential will measure how far the first remaining defect raises weight.

Only weights at most four are needed below. We may therefore do the entire calculation in the finite complex C/F^5C. It is finite-dimensional: a nonzero normalized tensor of weight below five has at most four slots. This quotient truncates the whole complex by total weight. It is not asserted to be the cobar complex of $k[[t]]/(t^5)$; that ideal need not define a quotient compatible with the coproduct.

5 The first correction and the cubic residual

The linear term has defect

$$Dt = -t \mid t.$$

It is primitive in the associated graded, but fails at weight two. To see whether a quadratic correction can remove the failure, calculate $D(t^2)$. This is the only coproduct multiplication we need to perform explicitly.

Put $u = t \otimes 1$, $v = 1 \otimes t$, and $r = t \otimes t$. Multiplicativity of Δ gives

$$\overline{\Delta}(t^2) = (u + v + r)^2 - u^2 - v^2 = 2uv + 2(u + v)r + r^2.$$

Since $-2 = 1$ in k, taking its negative gives

$$D(t^2) = t \mid t + t \mid t^2 + t^2 \mid t - t^2 \mid t^2.$$

In particular, $D_0(t^2) = t \mid t$. Adding t^2 therefore cancels the first defect:

$$\boxed{D(t + t^2) = c - v, \qquad c = t \mid t^2 + t^2 \mid t, \quad v = t^2 \mid t^2.}$$

The two remaining terms have different weights: c has weight three and v has weight four. The series $t + t^2$ satisfies the logarithm identity through total degree two. The question is whether a cubic correction can remove c.

Consider the weight-three part of the additive cobar complex:

$$k\,t^3 \xrightarrow{D_0} k\{t \mid t^2,\ t^2 \mid t\} \xrightarrow{D_0} k\,(t \mid t \mid t).$$

The three terms lie in cohomological degrees one, two, and three. Frobenius gives $D_0(t^3) = 0$. The graded product rule and $D_0(t^2) = t \mid t$ give

$$D_0(t \mid t^2) = -t \mid t \mid t, \qquad D_0(t^2 \mid t) = t \mid t \mid t.$$

The kernel in degree two is precisely $k\,c$, while the image from degree one is zero. Therefore

$$H^2(\mathrm{gr}_F^3 C) = k \cdot [c], \qquad [c] \neq 0.$$

A cubic correction is a scalar multiple of t^3. Its weight-three defect is zero, so it cannot change c. Terms of higher weight cannot change a cubic defect either. The attempted logarithm has reached its first unrepairable order.

In coordinates,

$$c(x, y) = xy^2 + x^2y, \qquad D(t + t^2)(x, y) = xy^2 + x^2y - x^2y^2.$$

The obstruction is the class of the leading term in the weight-three complex. The full defect is a coboundary—it is $D(t + t^2)$. There is no contradiction: the lifting problem prescribes a leading term, and the relevant boundaries are those available from corrections at the next weight.

6 The nonzero second differential

We can now identify the calculation as a spectral-sequence differential, including its target page. Use the convention

$$E_0^{s,q} = \mathrm{gr}_F^s C^{s+q}, \qquad E_1^{s,q} = H^{s+q}(\mathrm{gr}_F^s C, D_0).$$

The differential has bidegree

$$d_r : E_r^{s,q} \longrightarrow E_r^{s+r,q-r+1}.$$

It raises total cohomological degree by one and filtration degree by r. A representative whose lower-weight defects have been removed supplies its next differential by taking the leading remaining defect.[1]

Each next page is the cohomology of the preceding one: $E_{r+1} = H(E_r, d_r)$.

The one-cochain t gives a class in $E_1^{1,0}$. Its first possible differential is represented by

$$D_1 t = -t \mid t = -D_0(t^2).$$

Hence $d_1[t] = 0$. The explicit nullhomotopy is the correction t^2, and a representative appropriate to the next page is $t + t^2$. Its weight-three defect is c, so the proposed second differential is

$$d_2[t] = [c].$$

We must check that $[c]$ remains nonzero on E_2, rather than only on E_1. There are two checks.

First, no d_1 can hit it. The incoming group is

$$E_1^{2,-1} = H^1(\mathrm{gr}_F^2 C) = 0,$$

because the weight-two complex is the isomorphism

$$k\, t^2 \xrightarrow{D_0} k\,(t \mid t).$$

Second, its outgoing d_1 vanishes. Here the identity $D^2 = 0$ performs useful work. From $D(t+t^2) = c - v$, we obtain

$$Dc = Dv.$$

Taking the weight-four part gives

$$D_1 c = D_0 v.$$

The right side is explicitly

$$D_0(t^2 \mid t^2) = t \mid t \mid t^2 - t^2 \mid t \mid t.$$

Thus the first differential out of $[c]$ is a boundary. Its target group is nonzero; it is this displayed boundary that makes $[c]$ survive.[2] Since no incoming differential kills it, $E_2^{3,-1} = k \cdot [c]$.

The source also survives as $E_2^{1,0} = k \cdot [t]$: its only possible incoming d_1 comes from constants, whose differential is zero. We have proved

$$\boxed{d_2 : E_2^{1,0} \longrightarrow E_2^{3,-1}, \qquad [t] \longmapsto [t \mid t^2 + t^2 \mid t], \qquad d_2 \neq 0.}$$

Both spaces are one-dimensional, so this map is an isomorphism. The class with linear term t disappears on the next page. It cannot arise from a genuine primitive series with that leading term.

The index two measures the increase from weight one to weight three. It does not mean that the residual is a quadratic polynomial, or that it lies in cohomological degree three. The residual is cubic, and it is a two-cochain. Keeping these roles separate makes the page number intelligible.

This also clarifies the use of c. It is a cocycle for D_0, and it represents an E_2 class. It is not a cocycle for the full multiplicative differential: $Dc = Dv$. The filtered construction remembers precisely the correction needed to interpret its leading term.

7 The tower behind the pages

Let $K_s = C/F^sC$. Successive truncations fit into short exact sequences of complexes

$$0 \longrightarrow \operatorname{gr}_F^s C \longrightarrow K_{s+1} \longrightarrow K_s \longrightarrow 0.$$

They give exact triangles in the derived category of k-vector spaces,

$$\operatorname{gr}_F^s C \longrightarrow K_{s+1} \longrightarrow K_s \longrightarrow \operatorname{gr}_F^s C[1],$$

and connecting maps

$$\partial_s : H^1(K_s) \longrightarrow H^2(\operatorname{gr}_F^s C).$$

To compute ∂_s, choose a representative lift to K_{s+1} and apply D. Its differential lies in the kernel, which is the new weight layer. Another lift differs by a cochain in that layer, so its differential changes by a D_0-boundary. This proves that the obstruction class is independent of the chosen lift. Its vanishing is equivalent to the existence of a correction making the lift a cocycle.

Our calculation follows two consecutive stages:

Partial solution	Next residual	Obstruction class	Result
t in K_2	$-t \mid t$	Zero in $H^2(\operatorname{gr}^2 C)$	Correct by t^2 and lift to K_3
$t + t^2$ in K_3	c	Nonzero in $H^2(\operatorname{gr}^3 C)$	No lift to K_4

The spectral sequence organizes such successive lift-and-correct calculations, accounting for the ambiguity in earlier choices. In this example the quadratic correction is unique, and the second obstruction is the nonzero d_2 just computed. The weight-four calculation verifies that its value is defined on the second page.

There is a direct comparison with the edge-removal obstruction in *What a Boundary Map Can Prove*. There a connecting map measures whether local data extend across an edge. Here successive connecting maps measure whether an identity extends by one polynomial order. The complexes differ, but the operation is the same: lift a representative, take its differential, and identify the resulting class modulo changes of lift.

Formal-group obstruction calculations have long used degree filtrations in this way. Rezk's account of the Lubin–Tate theorem constructs a complex of symmetric cocycles, filters by polynomial degree, and calculates the resulting spectral sequence. His two-term symmetric-cocycle complex differs from the full normalized cobar complex used here; it provides a close setting for the method and its wider role.[3]

8 The obstruction becomes an addition law

The failed logarithm supplied a polynomial we know how to calculate. We can now reverse its job: use that polynomial as the correction in a new binary operation, and use its compatibility equation to check associativity. Its additive cocycle identity is

$$c(x,y)+c(x+y,z)=c(y,z)+c(x,y+z).$$

This is exactly $D_0c=0$, already proved in the weight-three complex. Define an operation on pairs by

$$(x,u)+(y,v)=\big(x+y,\ u+v-c(x,y)\big).$$

The two bracketings of a threefold sum have equal first coordinates. Their second coordinates differ by the two sides of the cocycle identity. They are therefore equal. Symmetry of c proves commutativity, $c(x,0)=0$ gives the identity $(0,0)$, and $c(x,-x)=0$ gives inverse $(-x,-u)$.

This is the addition law of length-two, 3-typical Witt vectors over k. To see the coefficient directly, first work over the integers with the two ghost polynomials

$$w_0(X,U)=X,\qquad w_1(X,U)=X^3+3U.$$

Requiring both ghost coordinates to add forces the second coordinate of a sum to be

$$U+V-\frac{(X+Y)^3-X^3-Y^3}{3}=U+V-X^2Y-XY^2.$$

The quotient is an integer polynomial. Reducing it modulo three gives the displayed addition law; no division by three is performed in k.[4]

The mixed term changes an operation we can test directly. The threefold sum in the new law is

$$[3](x,u)=(0,x^3).$$

Indeed, $c(x,x)=2x^3=-x^3$ and $c(2x,x)=0$, so the second coordinate of three copies is $3u-c(x,x)-c(2x,x)=x^3$. Componentwise addition on $\mathbf{G}_a^2$ would have zero three-series. The cubic correction changes the group structure even though the underlying space remains the affine plane.

The projection $(x,u)\mapsto x$ is a homomorphism to $\mathbf{G}_a$, with kernel the additive second coordinate. Thus c supplies the data of an extension. This construction gives a different group law; it does not produce the scalar logarithm ruled out in section 1. A polynomial that obstructed one compatibility problem now enforces associativity in another.

9 The finite correction procedure

The calculation can now be written as a procedure on polynomial coefficients. Suppose a trial coordinate satisfies the desired identity through total degree $N-1$. Compute its homogeneous defect $E_N(x,y)$ in degree N. Adding a_Nt^N changes that degree by

$$a_NC_N(x,y),\qquad C_N(x,y)=(x+y)^N-x^N-y^N.$$

The higher terms of $F(x, y) = x + y + xy$ do not affect this formula in degree N. To extend the identity one order, solve the finite coefficient equation

$$E_N + a_N C_N = 0.$$

At $N = 2$ in characteristic three, $E_2 = xy$ and $C_2 = -xy$, so $a_2 = 1$. At $N = 3$, the repaired defect is $-xy(x + y)$ and $C_3 = 0$. There is no available scalar correction. The first failed step has become visible in the matrix of a linear coefficient problem.

In a different law, a zero C_N accompanied by a zero defect would leave a free coefficient. That choice would have to remain available in later stages. A procedure that commits to arbitrary earlier coefficients can miss a later solution. The filtered-complex formulation keeps track of exactly this dependence and of when two correction choices represent the same obstruction.

For a fixed partial solution, two lifts to the next order have defects differing by a boundary in the new weight layer. The resulting obstruction class is independent of the choice of lift, and it vanishes exactly when that partial solution extends.

This also distinguishes two improvements to an ordinary derivative. Hasse coefficients detect a term that a derivative misses. The compatibility calculation decides whether those terms can belong to the requested change of coordinates. Better detection does not force solvability. Here it proves failure, locates its first order, and supplies a polynomial that we can reuse to define another associative addition law.

Two formal disks with the same first-order tangent space can have different residuals and later compatibility conditions because their multiplication laws differ. The coefficient equations retain that dependence on the original group law.

Notes and sources

1. The Stacks Project, Spectral sequences: filtered complexes, especially Lemma 12.24.2 for $E_0^{s,q} = \mathrm{gr}^s C^{s+q}$, the bidegree of d_r, and the representative formula; Lemma 12.24.3 identifies the first differential with a connecting map. The explicit weight-two through weight-four calculations above establish the claimed second differential without an infinite-convergence assertion. ↩

2. The target is $H^3(\mathrm{gr}_F^4 C)$. Set $u = t^2 \mid t \mid t$, $w = t \mid t^2 \mid t$, and $z = t \mid t \mid t^2$. Their D_0-images are respectively $t \mid t \mid t \mid t$, its negative, and itself, so the kernel is two-dimensional. In weight four, the degree-two basis is $t \mid t^3, t^2 \mid t^2, t^3 \mid t$; its images are $0, z-u, 0$. The quotient is therefore one-dimensional, represented for example by $u + w$. ↩

3. Charles Rezk, Notes on the Hopkins–Miller Theorem, §§5.1–5.8, printed pp. 7–10. The discussion constructs the formal-group symmetric-cocycle complex, filters it by polynomial degree, and computes its associated spectral sequence. The full normalized cobar complex and its particular characteristic-three calculation are presented directly here. ↩

4. Lars Hesselholt, Lecture notes on Witt vectors, opening construction of the Witt polynomials and their addition operations. For the first two 3-typical coordinates, the integer ghost-coordinate calculation in section 8 gives the required polynomial explicitly; the cocycle identity independently proves its associativity after reduction. ↩